In conclusion, I wish to express my most sincere thanks to Professor Sir J. J. Thomson, at whose suggestion this research was undertaken, and who has throughout given much encouragement by showing a never-failing interest in the progress of the work.

On a Class of Parametric Integrals and their Application in the Theory of Fourier Series.

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§ 1. The theorem commonly known as Parseval's Theorem, which, in its latest form, as extended by Fatou,* asserts that if $f(x)$ and $g(x)$ are two functions whose squares are summable, and whose Fourier constants are a_n, b_n and α_n, β_n, then the series

$$\tfrac{1}{2}a_0\alpha_0 + \sum_{n=1}(a_n\alpha_n + b_n\beta_n)$$

converges absolutely and has for its sum

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx,$$

must be regarded as one of the most important results in the whole of the theory of Fourier series.

I have recently, in the 'Proceedings' of this Society and elsewhere, had occasion to illustrate its usefulness, as well as that of certain analogous results to which I have called attention. They may be said, indeed, to have reduced the question of the convergence of Fourier series to the second plane. If we know that a trigonometrical series is a Fourier series, it is in a great variety of cases, embracing even some of the less usual ones, as well as those which ordinarily present themselves, all that we require. It has seemed to me, therefore, worth while to add another to the list of these results. This is the main object of the present paper, in which it is shown that if one of the functions has its $(1+p)$th power summable and the other its $(1+1/p)$th power summable, where p is any positive quantity, however small, then the above theorem is true with this modification, provided only the series in question is summed in the Cesaro way. In particular, the equality always holds in the ordinary sense whenever the series does not oscillate.

* Fatou, "Séries Trigonométriques et Séries de Taylor," 'Acta Mathematica,' 1905, vol. 30.

I have, however, treated this problem as a part of a larger one, and have shown that the cosine series whose typical coefficient is $(a_n\alpha_n + b_n\beta_n)$, and the sine series whose typical coefficient is $(a_n\beta_n - b_n\alpha_n)$ are, under the circumstances stated, the Fourier series of continuous functions.

I have also shown this to be the case when one of the functions is bounded and the other summable. When the bounded function has also bounded variation, they are the Fourier series of functions of bounded variation.

All the previously known results,* as well as that above stated, follow at once as simple corollaries. Moreover, the question of the convergence of the series

$$\tfrac{1}{2}a_0\alpha_0 + \sum_{n=1}(a_n\alpha_n + b_n\beta_n) \quad \text{and} \quad \sum_{n=1}(a_n\beta_n - b_n\alpha_n),$$

as well as of other series obtained by ascribing particular values to x in the Fourier series in question, is made to depend on the examination of certain parametric integrals at the point in question.†

§ 2. We first prove an auxiliary theorem in the theory of sets of points.

Theorem.—*If* G *is a closed set of points of positive content* E *in the interval* (a, b), *and* G_x *is the set got by simply translating the set* G *a distance* x *in the direction* a *to* b, *then the sets* G *and* G_x *have a common closed set of points, whose content differs from* E *by less than an assigned quantity* e, *provided only* x *is less than a quantity* X *which can be specified.*

Suppose the black intervals of the closed set G arranged in any order, $d_1, d_2 \ldots, d_n \ldots$

Let e' be any chosen positive quantity less than e, and let R_m denote the sum of all but the first m of the black intervals, where m is the first integer for which

$$R_m < e'. \tag{1}$$

We then take X to be the largest quantity satisfying the two inequalities

$$X \leq (e - e')/(1+m), \tag{2}$$
$$X \leq d_n, \qquad (n \leq m). \tag{3}$$

* For these results see my paper "On the Integration of Fourier Series," 'Lond. Math. Soc. Proc.,' 1910, Series 2, vol. 9, pp. 449–462.

† The method, as has been kindly pointed out to me, is not quite new. A particular case of our parametric integral was employed by A. C. Dixon in 1909, for the purpose of proving Fatou's Theorem, in a paper entitled "On a Property of a Summable Function," 'Camb. Phil. Soc. Proc.,' vol. 15, pp. 210—6. In reference to this proof it may be noted that no use is there made of the property of a Fourier series of never diverging properly, provided the Fourier function is continuous. In consequence the assumption has to be made that the series $\Sigma(a_n^2 + b_n^2)$ is known to converge to a sum not greater than the quantity to which it is to be proved equal. I notice that in Dr. Dixon's ingenious proof of the continuity of the parametric integral considered by him, the tacit assumption is made that Schwarz's inequality is true of Lebesgue improper integrals, a result which had not at that time been proved.

Now, by reason of the inequality (3), when we translate the interval (a, b) carrying the set G and its black intervals, each of the black intervals d_n, with index not greater than m, will have in common with itself, after translation, a part of length $(d_n - x)$, x denoting the amount of translation. Thus there will be at least m such common parts of intervals containing no point of G either before or after the translation. Hence, if we denote by I the content of the closed set, consisting of all the points of G before and after translation,

$$\begin{aligned} \mathrm{I} &\leq b-a+x-\sum_1^m (d_n - x), \\ &\leq b-a+x+mx-\sum_1^m d_n, \\ &\leq \mathrm{E}+\mathrm{R}_m+(1+m)\,x, \\ &< \mathrm{E}+e, \end{aligned} \tag{4}$$

using (1) and (2).

Now, if I_1 and I_2 are the content of two closed sets, I that of the set consisting of both sets, and I′ that of their common part, we have*

$$\mathrm{I}_1+\mathrm{I}_2 = \mathrm{I}+\mathrm{I}'. \tag{5}$$

In our case $\mathrm{I}_1 = \mathrm{I}_2 = \mathrm{E}$. Hence by (4) and (5),

$$\mathrm{I}' > \mathrm{E}-e. \quad \text{(Q.E.D.)}$$

Cor.—*The same is true if, in the above enunciation, we omit throughout the word "closed."*

For, in this case, we can find a closed component of G, say G′, whose content differs from E by less than $\frac{1}{2}e$. Also, by the theorem, we can determine X so that, for any value of x less than X, the common points of G′ and G'_x, *a fortiori* the common points of G and G_x, form a set whose content differs from that of G′ by less than $\frac{1}{2}e$, and therefore from that of G by less than e.

§ 3. The necessary and sufficient condition that a function $\mathrm{F}(x)$ should be an integral† has been stated by Vitali in a form which it will be convenient to modify slightly for the purposes in hand. From Vitali's condition we know that, if $f(x)$ is a summable function, and e a chosen positive quantity, we can find E so that, if D denote any set of non-overlapping intervals of content less than E, $\int_{\mathrm{D}} f(x)\,dx$, which is the increment of the integral over the set of intervals D, is numerically less than e. But we also know‡ that,

* Young's 'Theory of Sets of Points,' Cambridge University Press, 1906, p. 84.

† See my paper on "Semi-integrals and Oscillating Successions of Functions," 'Lond. Math. Soc. Proc.,' 1910, Series 2, vol. 9, p. 291, *seq.*, where also references will be found.

‡ *Loc. cit.*, p. 291.

if G is a measurable set of points of content less than E, we can find a sequence of sets of intervals $D_1, D_2, \ldots, D_n, \ldots$ having the points of G as common points, and no other common points, except possibly a set of content zero, and that $\int_G f(x)\,dx$ is the unique limit of $\int_{D_n} f(x)\,dx$, when n increases indefinitely. Since the content of D_n has that of G as limit, it must be less than E, when n is large enough; thus, by Vitali's property, $\int_{D_n} f(x)\,dx$ becomes eventually less than e, and therefore its unique limit, $\int_G f(x)\,dx$, is also less than e. Thus we have the following property of a summable function, which we shall need in the sequel:—

If $f(x)$ is a summable function, and e a chosen positive quantity, we can find E, *so that the integral of $f(x)$ over any and every set of points of content less than* E *is less than e.*

§ 4. Theorem.—*If $f(t)$ is a summable function, and* G *a set of positive content, $\int_G f(x+t)\,dt$ is a continuous function of x.**

For if, as in § 2, G_x denote the set got by translating the set G a distance x, and I' denote the common part of the sets G and G_x, we have seen that the remaining components E and E_x of G and G_x have content less than e, provided $|x|$ is less than a certain quantity X, which can be specified. Moreover, by the fundamental property of an integral, the quantity e, here used, may be so chosen that the integral of $f(t)$ over any and every set of content less than e is numerically less than a positive quantity e' chosen at will. This being so, we have

$$\int_E f(t)\,dt < e', \qquad \int_{E_x} f(t)\,dt < e',$$

while
$$\int_G f(t)\,dt = \int_{I'} f(t)\,dt + \int_E f(t)\,dt,$$

and
$$\int_G f(x+t)\,dt = \int_{G_x} f(t)\,dt = \int_{I'} f(t)\,dt + \int_{E_x} f(t)\,dt.$$

Hence
$$\left| \int_G f(x+t)\,dt - \int_G f(t)\,dt \right| < 2e',$$

for all values of x numerically less than X.

This shows that, as x approaches zero, all possible limits of $\int_G f(x+t)\,dt$

* This theorem and that of the next article might have deduced from a theorem, to which my attention has been called, proved by Lebesgue on pp. 15 and 16 of his 'Leçons sur les Séries Trigonométriques.'

differ from $\int_G f(t)\,dt$ by, at most, $2e'$, and therefore all coincide with $\int_G f(t)\,dt$, since e' may be as small as we please. Thus, $\int_G f(x+t)\,dt$ is continuous with respect to x at $x = 0$.

Writing $a+x$ for x, the same argument proves that the integral is continuous at $x = a$. Hence the integral is a continuous function of x, as was to be proved.

§ 5. Theorem.—*If $f(t)$ is a summable function of t, and $g(t)$ is a bounded function,*

$$\int_a^b f(t+x)\,g(t)\,dt,$$

is a continuous function of x.

Case 1.—Let $f(t) \geq 0$.

Divide the whole range (K, K') of values of $g(t)$ into n equal parts, and let $H_0, H_1, H_2, \ldots, H_n$, be the corresponding sets of points, so that, for instance, H_r is the set of points at which

$$K + r(K'-K)/n \leq g(t) < K + (r+1)(K'-K)/n. \tag{1}$$

Also let $g_r(t)$ denote the function which is zero except at the points of H_r, and at the points of that set has as value the greatest of the three quantities given in the inequality (1). Then at the points of H_r the new function $g_r(t)$ differs from $g(t)$ by at most $(K'-K)/n$.

Now $$\int_a^b f(x+t)\,g_r(t)\,dt = [K+(r+1)(K'-K)/n]\int_{H_r} f(x+t)\,dt,$$

so that, by the preceding theorem, this integral is a continuous function of x. Hence also

$$\sum_0^n \int_a^b f(x+t)\,g_r(t)\,dt = F(x, n)$$

is a continuous function of x. Moreover, by what was already pointed out, it is greater than $\int_b^a f(x+t)\,g(t)\,dt$ by at most $(K'-K)\int_a^b f(x+t)\,dt/n$, that is by less than $(K'-K)\,M/n$, where M is a fixed quantity, since $\int_a^b f(x+t)\,dt$ is a continuous function of x, and has, therefore, finite upper and lower bounds. Hence, as n increases indefinitely, $F(x, n)$ converges uniformly to

$$\int_a^b f(x+t)\,g(t)\,dt$$

as unique limiting function, and therefore that limiting function is a continuous function of x.

Case 2.—Let $f(t)$ be sometimes positive, sometimes negative.

Then $f(t)$ is the difference of two positive summable functions, one of which is equal to $f(t)$ except where $f(t)$ is negative, where the new function is zero. In this way our integral can be expressed as the difference of two integrals each of which is a continuous function of x, by Case 1, so that in this case also the integral is a continuous function of x.

Thus in every case the theorem is true.

Cor.—*If $g(t)$ is only bounded below (above), the integral in question is a lower (upper) semi-continuous function of x in the extended sense, the value infinity above (below) being admitted, provided $f(t)$ is positive, or bounded below.*

If $f(t)$ is negative, or bounded above, and $g(t)$ is bounded below (above), the integral is an upper (lower) semi-continuous function of x.

For, supposing $f(t)$ to be positive, and denoting by $g_k(t)$ the function equal to $g(t)$ wherever $g(t) \leq k$ and elsewhere zero, we know, by the definition of the integral that the integral in question is the unique limit when k increases indefinitely of the integral

$$\int_a^b f(x+t)\, g_k(t)\, dt.$$

This latter integral is, by the above theorem, a continuous function of x, and it is a monotone increasing function of k, so that the limiting function is a lower semi-continuous function of x. This proves the first of the alternative theorems; similarly the others may be proved.

§ 6. Theorem.—*If $f(t)$ and $g(t)$ are such that their squares are summable, then*

$$\int_b^a f(x+t)\, g(t)\, dt$$

is a continuous function of x.

Let Q denote the content of the set of those points at which $g(t)$ is numerically greater than k. Then, since the square of $g(t)$ is summable, we can choose k so large that

$$\int_Q [g(t)]^2\, dt \leq e,$$

where e is any chosen positive quantity.

But, since the squares of $f(t)$ and $g(t)$ are summable, we have, by Schwarz's inequality,

$$\begin{aligned}\left[\int_Q f(t+x)\, g(t)\, dt\right]^2 &\leq \int_Q [f(t+x)]^2\, dt \int_Q [g(t)]^2\, dt \\ &\leq e \int_Q [f(t+x)]^2\, dt \\ &\leq e\mathrm{M},\end{aligned}$$

where M is the upper bound of the continuous function of x represented by $\int_Q [f(t+x)]^2\,dt$.

Hence

$$\int_a^b f(x+t)\,g(t)\,dt = \int_a^b f(x+t)\,g_k(t)\,dt + \int_Q f(t+x)\,g(t)\,dt$$
$$= \mathrm{G}(x, k) + \sqrt{e'}\mathrm{M},$$

where $g_k(t) = g(t)$, wherever it is numerically $\leq k$, and elsewhere $g_k(t) = 0$, so that, by the preceding theorem, $\mathrm{G}(x, k)$ is a continuous function of x, and e' is numerically $\leq e$.

This shows that $\mathrm{G}(x,k)$ converges uniformly to the integral on the left, which is therefore a continuous function of x.

§ 7. The preceding theorem is a special case of the following, in which p represents any positive quantity (not zero), rational or irrational.

Theorem.—*If $f(t)$ and $g(t)$ are such that $[f(t)]^{1+p}$ and $[g(t)]^{1+1/p}$ are summable, then*

$$\int_a^b f(x+t)\,g(t)\,dt$$

is a continuous function of x.

For since the arithmetic mean is not less than the geometric mean, we have, for all positive integers r and s,

$$(u^r v^s)^{1/(r+s)} \leq (ru+sv)/(r+s).$$

Putting $u = \mathrm{U}^{(r+s)/r}$, $v = \mathrm{V}^{(r+s)/s}$, $sp = r$, this gives

$$(1+p)\,\mathrm{UV} \leq p\mathrm{U}^{1+1/p} + \mathrm{V}^{1+p}. \tag{1}$$

Since this is true for every positive rational value of p, it is also true for every positive irrational value of p.

Hence, if $f(t)$ and $g(t)$ are positive functions,

$$(1+p) f(x+t)\,g(t) \leq p\,[g(t)]^{1+1/p} + [f(x+t)]^{1+p}.$$

Integrating over any set Q,

$$(1+p)\int_Q f(x+t)\,g(t)\,dt \leq p\int_Q [g(t)]^{1+1/p}\,dt + \int_Q [f(x+t)]^{1+p}\,dt$$
$$\leq p\int_Q [g(t)]^{1+1/p}\,dt + \int_{Q_x} [f(t)]^{1+p}\,dt, \tag{2}$$

where Q_x is the set got from Q by translation through a distance x.

Now let Q denote the set of those points at which $[g(t)]^{1+1/p}$ is numerically greater than k, where k is chosen so large that the content of Q is less than an assigned small positive quantity e'; this is possible, since $[g(t)]^{1+1/p}$ is summable.

Moreover, by the fundamental property of an integral, e' could be chosen so small that the integrals of $[g(t)]^{1+1/p}$ and $[f(t)]^{1+p}$ over any set of content less than e' are less than e, where e is a small positive quantity chosen previous to all.

In this way, since Q_x has the same content as Q, each of the integrals on the right-hand side of (2) is numerically less than e. Thus

$$(1+p)\int_Q f(x+t)\,g(t)\,dt \leqq (p+1)\,e.$$

Hence

$$\int_a^b f(x+t)\,g(t)\,dt = \int_a^b f(x+t)\,g_k(t)\,dt + \int_Q f(x+t)\,g(t)\,dt = G(x,k)+e''.$$

where e'' is $\leqq e$ and $G(x, k)$ is a continuous function of x, since $g_k(t)$ denotes the function $= g(t)$ wherever it is $\leqq k$, and $g(t)$ is zero elsewhere. Hence, $G(x, k)$ converges uniformly as k increases indefinitely to the integral under consideration, which is therefore a continuous function of x.

This proves the theorem in the case when $f(t)$ and $g(t)$ are positive functions. Hence, changing the sign of either or both the functions, it at once follows that the theorem is true, provided $f(t)$ and $g(t)$ retain their sign. But in the general case we have only to split up f into f_1+f_2 and g into g_1+g_2 in the usual way, where f_1 and g_1 are positive and f_2 and g_2 are negative, to express our integral as the sum of four integrals, each of which has already been proved continuous. Thus in the general case also the integral under consideration is a continuous function of x, which proves the theorem.

§ 8. Theorem.—*If $f(t)$ is summable and $g(t)$ is bounded, we may reverse the order of repeated integration and write*

$$\int_a^b dx \int_c^d f(x+t)\,g(t)\,dt = \int_c^d dt \int_a^b f(x+t)\,g(t)\,dx.$$

First let f be a positive function.

Then since the repeated integrals of $f(x+t)$ evidently exist and are equal, we may, by a known theorem,* reverse the order of integration.

Next let f be not always positive.

Then we can write

$$f(t) = f_1(t) - f_2(t),$$

where f_1 and f_2 are both positive functions, f_1 being equal to f wherever $f \geqq 0$, and being zero elsewhere. Expressing our integral as the difference of two integrals, one involving f_1 and the other f_2, the required result follows,

* W. H. Young, "On Change of Order of Integration in an Improper Repeated Integral," 'Camb. Phil. Soc. Trans.,' 1910, vol. 21, p. 364.

since, by what has been already remarked, we may reverse the order of integration in each of the two auxiliary integrals.

§ 9. Theorem.—*If* $f(t)$ *and* $g(t)$ *are such that* $[f(t)]^{1+p}$ *and* $[g(t)]^{1+1/p}$ *are summable, then we may reverse the order of repeated integration, and write*

$$\int_a^b dx \int_c^d f(x+t)\, g(t)\, dt = \int_c^d dt \int_a^b f(x+t)\, g(t)\, dx.$$

As in the proof of the theorem of § 7,

$$\int_a^b f(x+t)\, g(t)\, dt = \int_a^b f(x+t)\, g_k(t)\, dt + e'', \tag{1}$$

where $e'' < e$, and g_k is a bounded function of t, having $g(t)$ as limit when k increases indefinitely. Since, by the theorem quoted, the left-hand side of this equation is a continuous function of x, and the same is true of the integral on the right, while e'' is bounded, we may integrate with respect to x, and get

$$\int_c^d dx \int_a^b f(x+t)\, g(t)\, dt = \int_c^d dx \int_a^b f(x+t)\, g_k(t)\, dt + \int_c^d e''\, dx.$$

Now $g_k(t)$ is a bounded function, therefore, by the preceding theorem, we may reverse the order of integration in the first integral on the right in the preceding equation. We thus get

$$\left| \int_c^d dx \int_a^b f(x+t)\, g(t)\, dt - \int_a^b dt \int_c^d f(x+t)\, g_k(t)\, dt \right| < e(d-c). \tag{2}$$

Letting e diminish down to zero as limit, k increases without limit, so that by de la Vallée Poussin's definition of an improper integral, $\int_c^d f(x+t)\, g_k(t)\, dt$ has as limit $\int_c^d f(x+t)\, g(t)\, dt$. Thus, by (2) we get in the limit

$$\int_c^d dx \int_a^b f(x+t)\, g(t)\, dt - \int_a^b dt \int_c^d f(x+t)\, g(t)\, dt = 0.$$

This proves the theorem.

Cor.—*The theorems of* §§ 8, 9, *remain true if we introduce into the integrand any bounded factor* $h(x, t)$.

For, in the first place, if $f(t)$ and $g(t)$ are positive functions, the corollary is an immediate consequence of the known theorem already quoted. If $f(t)$ and $g(t)$ are not positive functions, we may write $f = f_1 - f_2$, and $g = g_1 - g_2$, where f_1, f_2, g_1 and g_2 are positive functions, while f_1 and f_2 both enjoy the same property as to summability as f, and g_1 and g_2 as g. Thus the corollary holds when the integrand is any one of the four products

$$f_i(x+t)\, g_j(t)\, h(x, t),$$

i and j being either 1 or 2. By suitable additions and subtractions it at once follows that the corollary is true.

§ 10. In the following theorems the functions f and g are supposed to be summable, and we consider their Fourier series. In the integrals between the limits of integration $(-\pi, \pi)$ which occur, f and g are supposed to be rendered periodic in the usual way, by attributing outside the interval $(-\pi, \pi)$ the values in that interval periodically. The properties of f and g hypothecated as to boundedness or summability are, of course, unaffected by this convention. We now have, however, by a simple change of variable,

$$\int_{-\pi}^{\pi} f(x+t)\, g(t)\, dt = \int_{-\pi+x}^{\pi+x} f(t)\, g(t-x)\, du = \int_{-\pi}^{\pi} f(t)\, g(t-x)\, dt,$$

$$\int_{-\pi}^{\pi} f(t-x)\, g(t)\, dt = \int_{-\pi-x}^{\pi-x} f(t)\, g(t+x)\, du = \int_{-\pi}^{\pi} f(t)\, g(t+x)\, dt,$$

so that

$$\int_{-\pi}^{\pi} [f(t+x) \pm f(t-x)]\, g(t)\, dt = \pm \int_{-\pi}^{\pi} [g(t+x) \pm g(t-x)]\, f(t)\, dt. \qquad \text{(A)}$$

Thus, for instance, in applying the preceding theorems, the hypotheses with respect to f and g may be interchanged without affecting the result.

It should also be remarked that, owing to the periodicity of the functions which occur in the integrands, the functions denoted by the integrals are themselves periodic, so that at the extremities of the interval $(-\pi, \pi)$ their values agree. Thus the continuity of the integrals at these extremities has the special character so often required in the applications to Fourier series.

§ 11. Theorem.—*If $f(t)$ and $g(t)$ are such that* $[f(t)]^{1+p}$ and $[g(t)]^{1+1/p}$ *are summable, or if $f(t)$ is summable and $g(t)$ bounded, and if the Fourier series of f and g are*

$$f(t) \sim \tfrac{1}{2} a_0 + \sum_{n=1} (a_n \cos nt + b_n \sin nt)$$

$$g(t) \sim \tfrac{1}{2} \alpha_0 + \sum_{n=1} (\alpha_n \cos nt + \beta_n \sin nt),$$

then

$$\tfrac{1}{2} a_0 \alpha_0 + \sum_{n=1} (a_n \alpha_n + b_n \beta_n) \cos nt$$

is a Fourier series and the corresponding function is

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} [f(t+x) + f(t-x)]\, g(t)\, dt.*$$

For, in either of the cases specified, we can, as we have seen (§§ 8, 9), reverse the order of integration in the repeated integral

$$\int_a^b dt \int_c^d f(t+x)\, g(t) \cos nx\, dx.$$

* In this integral f and g must be supposed to have been made periodic in the usual way, see § 10.

Hence, denoting the function given at the end of the enunciation by $q(x)$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} q(x)\cos nx\,dx = \frac{1}{2\pi^2}\int_{-\pi}^{\pi}dt\int_{-\pi}^{\pi}f(t+x)\,g(t)\cos nx\,dx$$
$$+\frac{1}{2\pi^2}\int_{-\pi}^{\pi}dt\int_{-\pi}^{\pi}f(t-x)\,g(t)\cos nx\,dx$$
$$= \frac{1}{2\pi^2}\int_{-\pi}^{\pi}dt\int_{-\pi}^{\pi}f(u)\,g(t)\cos n(u-t)\,du$$
$$+\frac{1}{2\pi^2}\int_{-\pi}^{\pi}dt\int_{-\pi}^{\pi}f(u)\,g(t)\cos n(t-u)\,du,$$

where, in the changes of variable here used, the limits of integration may still be taken to be $-\pi$ and π, by reason of the periodicity of the integrands. Hence the integral on the left

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}f(u)\cos nu\,du\,\frac{1}{\pi}\int_{-\pi}^{\pi}g(t)\cos nt\,dt+\frac{1}{\pi}\int_{-\pi}^{\pi}f(u)\sin nu\,du\,\frac{1}{\pi}\int_{-\pi}^{\pi}g(t)\sin nt\,dt$$
$$= a_n\alpha_n+b_n\beta_n.$$

Since $\int_{-\pi}^{\pi} q(x)\sin nx\,dx$ is obviously zero, this proves the theorem.

Cor.—*The above theorem remains true in all cases in which the reversal of the order of partial differentiation used at the beginning of the proof is allowable.*

§ 12. Theorem.—*If $f(t)$ and $g(t)$ are such that $[f(t)]^{1+p}$ and $[g(t)]^{1+1/p}$ are summable, or if $f(t)$ is summable and $g(t)$ bounded, and if the Fourier constants of f are a_n, b_n, and those of g are α_n and β_n, then*

$$\sum_{n=0}(a_n\beta_n-b_n\alpha_n)\sin nx$$

is a Fourier series, and the corresponding function is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(t+x)-f(t-x)]\,g(t)\,dt.$$

The proof is on precisely the same lines as that of the last theorem. We merely have to interpret $q(x)$ as the function just given, and replace the cosines by sines, and we get

$$\frac{1}{\pi}\int_{-\pi}^{\pi} q(x)\sin nx\,dx = a_n\beta_n-b_n\alpha_n.$$

Since $\int_{-\pi}^{\pi} q(x)\cos nx = 0$, this proves the theorem.

§ 13. Theorem.—*If*

$$f(t) \backsim \tfrac{1}{2}a_0 + \sum_{n=1} (a_n \cos nt + b_n \sin nt), \tag{1}$$

and

$$g(t) \backsim \tfrac{1}{2}\alpha_0 + \sum_{n=1} (\alpha_n \cos nt + \beta_n \sin nt), \tag{2}$$

then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(t+x)+f(t-x)]\, g(t)\, dt = \tfrac{1}{2}a_0\alpha_0 + \sum_{n=1} (a_n\alpha_n + b_n\beta_n) \cos nx, \tag{3}$$

and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(t+x)-f(t-x)]\, g(t)\, dt = \sum_{n=1} (a_n\beta_n - b_n\alpha_n) \sin nx, \tag{4}$$

for all values of x in each of the two following cases:—

(i) *$f(t)$ is summable and $g(t)$ a function of bounded variation;*

(ii) *the squares of $f(t)$ and $g(t)$ are summable functions;*

and for every value of x for which the last series has a unique limit (sum), when $[f(t)]^{1+p}$ and $[g(t)]^{1+1/p}$ are summable functions.

For in all these cases, as has been proved, the series (3) and (4) are Fourier series, and the integrals on the left are the corresponding Fourier functions, so that the statements (3) and (4) are true if we replace the sign $=$ by $\sim$.

But the integrals on the left of (3) and (4) have been shown to be continuous functions of x in all the cases here considered. Hence, by a known property of Fourier series, the series (3) and (4) cannot diverge properly, and they represent their Fourier functions, except at points where they oscillate. This proves the final statement. It remains to prove that, in cases (i) and (ii), the series (3) and (4) do not oscillate.

In case (i) we use the equality (A) of § 10

$$\int_{-\pi}^{\pi} [f(t+x) \pm f(t-x)]\, g(t)\, dt = \pm \int_{-\pi}^{\pi} [g(t+x) \pm g(t-x)]\, f(t)\, dt, \tag{A}$$

to show that each of these functions of x has bounded variation. For the property of being monotone is obviously preserved in parametric integration. Hence, the integrand of the second of the integrals in (A) being in our case a function of x of bounded variation, that integral, and therefore also the other integral, represents a function of x of bounded variation.

But the Fourier series of a function of bounded variation always represents that function. This, therefore, proves the theorem in case (i).

In case (ii) both $[f(t)]^2$ and $[g(t)]^2$ are summable, therefore, putting $p = 1$, $x = 0$, in the result already proved in our final case, and, taking f and g to be the same, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(t)]^2\, dt = \tfrac{1}{2}a_0^2 + \sum_{n=1} (a_n^2 + b_n^2), \tag{5}$$

where we have introduced the sign of equality, since the series on the right, being a series of positive terms, cannot oscillate.

Similarly
$$\frac{1}{\pi}\int_{-\pi}^{\pi}[g(t)]^2\,dt = \tfrac{1}{2}\alpha_0^2 + \sum_{n=1}(\alpha_n^2+\beta_n^2), \tag{6}$$

Hence the series (3) and (4) converge for all values of x, since each of the terms is numerically less than, or equal to, the sum of the corresponding terms of the convergent series (5) and (6). Thus, by what was already proved for our final case, the theorem holds in case (ii).

Thus the complete theorem is proved.

Cor. 1.—*In cases* (i) *and* (ii) *the convergence of the series* (3) *is uniform.*

For the Fourier series of a continuous function of bounded variation converges uniformly, while in the above proof, in case (ii), the test for the convergence is Weierstrass's test for uniform convergence.

Cor. 2.—*In the final case of the preceding theorem, the equalities* (3) *and* (4) *hold for all values of x, provided the summation of the series be performed in the Cesaro manner, and the series converge then uniformly.*

For a Fourier series converges, and converges uniformly, at every point at which the function is continuous, when summed in the Cesaro way.

§ 14. In the preceding article only a few of the consequences of the fact that the trigonometrical series whose general terms are respectively

$$(a_n\alpha_n+b_n\beta_n)\cos nx \quad\text{and}\quad (a_n\beta_n-b_n\alpha_n)\sin nx$$

are, under the various circumstances specified in § 13, the Fourier series of continuous functions have been deduced. For the purposes of application the mere fact that these series are Fourier series will often suffice, quite irrespective of whether they converge or not. Moreover, from the fact that the functions to which they belong are continuous it follows that their allied series, viz., those whose general terms are

$$(a_n\alpha_n+b_n\beta_n)\sin nx \quad\text{and}\quad (a_n\beta_n-b_n\alpha_n)\cos nx$$

are Fourier series.

In fact, if we denote for brevity the functions of which the two series first mentioned are the Fourier series by $q_1(x)$ and $q_2(x)$, then the two latter series have for their Fourier functions

$$\frac{1}{\pi}\frac{d}{dx}\int_0^{\pi}[q_1(x+t)+q_1(x-t)]\log\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}t\,dt$$

and
$$\frac{1}{\pi}\frac{d}{dx}\int_0^{\pi}[q_2(x+t)+q_2(x-t)]\log\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}t\,dt$$

respectively.

This is, indeed, an immediate consequence of the formula (IV) on p. 19 of my recent paper " On the Fourier Constants of a Function."*

§ 15. In certain cases these two new functions are also plainly continuous. A knowledge, however, of $q_1(x)$ and $q_2(x)$ alone will often suffice to determine the question of the convergence of the allied series, as well as of the series to which they themselves belong. It follows, in fact, from § 10 of the paper just quoted,† and other known results, that if $q_1(x)$, for example, has all its derivates finite, not necessarily bounded, then both the series to which it belongs and the allied series will converge everywhere; they will, moreover, converge at an isolated point at which all the four derivates are finite.

We may sum up all these and other facts of a similar nature by saying that the whole question as to the convergence, or other properties, of the four series, whether for all values, or for particular values, of the variable x, is reducible to a discussion of the properties of certain parametric integrals. Moreover, sufficient indications have been given as to the properties of these integrals in a large and important class of cases.

§ 16. *Added May* 15.—So far we have only considered ordinary Fourier series. The method of the paper may, however, be applied to generalised Fourier series. We may thus, for example, obtain a second proof of the theorem of § 9 of my paper quoted in § 1.

It should also be remarked that the whole question of the convergence of the series $\Sigma(a_n\alpha_n+b_n\beta_n)$ and $\Sigma(a_n\beta_n-b_n\alpha_n)$ in the most general case in which the coefficients refer to ordinary Fourier series may be shown to turn on the properties of the functions

$$\frac{d}{dx}\left\{\frac{1}{\pi}\int_{-\pi}^{\pi}[\mathrm{F}(t+x)-\mathrm{F}(t-x)]\,g(t)\,dt\right\}$$

and
$$\frac{d}{dx}\left\{\frac{1}{\pi}\int_{-\pi}^{\pi}[\mathrm{F}(t+x)+\mathrm{F}(t-x)]\,g(t)\,dt\right\}.$$

The Fourier series of these functions, which may be proved to exist except for a set of values of x of content zero, are easily seen to be $\Sigma(a_n\alpha_n+b_n\beta_n)\cos nx$ and $\Sigma(a_n\beta_n-b_n\alpha_n)\sin nx$. Here $f(t)$ and $g(t)$ may be any summable functions, and $\mathrm{F}(t)$ denotes $\int_0^t f(t)\,dt$.

* 'Roy. Soc. Proc.,' 1911, A, vol. 85, pp. 14—24. In this formula IV on p. 19, as well as in the last five formulæ on p. 21, the factor $1/\pi$ has been omitted on the left-hand side.

† It may be noted that the result in question may also be obtained by actual summation of the first n terms of the allied series, and the use of a theorem of Riemann-Lebesgue.